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Wigner Functions and Tomograms of the Klauder-Perelomov Coherent States for the Pseudoharmonic Oscillator

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Abstract Using the coherent state representation of Wigner operator and the technique of integration within an ordered product (IWOP) of operators, the Wigner functions of the Klauder-Perelomov coherent states (KP-CSs) for the pseudoharmonic oscillator (PHO) are obtained and the variations of the Wigner functions with the parameters k and z are discussed. Moreover, the tomograms of the KP-CSs for the PHO are calculated by virtue of intermediate coordinate-momentum representation in quantum optics.

Keywords KP-CSs · IWOP technique · Wigner function · Tomogram

1 Introduction

It is well known that phase space techniques have been proved to be very effective tools in various branches of physics. Among various phase space, the Wigner distribution functions [1, 3, 12] are used popularly since two marginal distributions of the Wigner function lead to measuring probability in coordinate space and momentum space respectively. Experimentally, various photon states have been generated by the micromaser [2, 11] and the Wigner functions of some cavity fields can be measured in the whole phase space by a scheme based on interaction between cavity fields and atoms [15, 17]. Physically, the Wigner distribution function describes a signal in space and frequency simultaneously and it can be considered as the local frequency of the signal [16]. Recently, it was found that using the optical tomography was possible to measure the quantum state. A generalization of the optical tomography method leads to introducing the tomogram, which can be considered as a new representation of quantum mechanics. This tomogram can be represented in the form of an integral transform of the wave function of the quantum state, and the new representation of

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quantum states has been used to analyze the probability [8]. Therefore, it is significant to construct and measure the Wigner functions and the tomograms of quantum states by using some theoretical schemes.

In this paper, the Wigner functions and tomogram functions of the Klauder–Perelomov coherent states (KP-CSs) for the pseudoharmonic oscillator (PHO) are given. The reason why we pay attention to this case is that there are three aspects. Firstly, the PHO potential is an anharmonic potential, which like the usual harmonic oscillator potential (HOP), also allows an exact mathematical treatment. This potential may be considered in a certain sense as an intermediate potential between the HOP (an ideal model) and anharmonic potentials (which are more realistic) [9]. Secondly, a comparative analysis of the 3-dimensional HOP and PHO is performed in [10]. Thirdly, the coherent states are of special importance due to their remarkable mathematical properties and interesting physical applications in quantum optics.

The KP-CSs for the PHO [9] are obtained if the generalized displacement unitary operator $\exp(\beta K_+ - \beta^* K_-)$ on the lowest state of the quantum system $|n=0,k\rangle$

$$|z,k\rangle_{KP} = \exp\left(\beta K_+ - \beta^* K_-\right)|0,k\rangle = e^{zK} + e^{\Gamma K_3} e^{-z^* K_-}|0,k\rangle\,, \tag{1}$$

where $\beta = -\frac{1}{2}\theta e^{-i\varphi}$, $z = \frac{\beta}{|\beta|}\tanh|\beta|$ and the group generator $K_\pm$ and K_3 are satisfied the following relation

$$K_3 = \frac{1}{2}\left[K_-, K_+\right], \tag{2}$$

where $\Gamma = \ln(1-|z|^2)$ is Euler's gamma function, k is Bargmann index, the parameters $\theta \in (-\infty,\infty)$ and $\varphi \in [0,2\pi]$ are group parameters similar to the Euler angles. The condition $|z|<1$ shows that the SU(1, 1) KP-CSs $|z,k\rangle_{KP}$ are defined in the interior of the unit disc. In terms of the basis vectors $|n\rangle$ in the Hilbert space, the KP-CSs of the PHO may be expanded as [9]

$$|z,k\rangle_{KP} = N\left(|z|^2\right)\sum_{n=0}^{\infty}\frac{z^n}{\sqrt{\rho\,(n,k)}}|n\rangle\,, \tag{3}$$

where

$$\rho\,(n,k) = \frac{\Gamma(n+1)\Gamma(2k)}{\Gamma(n+2k)}, \tag{4}$$

and the normalization constant $N(|z|^2)$ is obtained from the normalization condition ${}_{KP}\langle z,k|z,k\rangle_{KP} = 1$, so that

$$N(|z|^2) = (1-|z|^2)^k. \tag{5}$$

Therefore, the normalized KP-CSs for the PHO have been defined as

$$|z,k\rangle_{KP} = (1-|z|^2)^k\sum_{n=0}^{\infty}\frac{z^n}{\sqrt{\frac{\Gamma(n+1)\Gamma(2k)}{\Gamma(n+2k)}}}|n\rangle\,. \tag{6}$$

2 The Wigner Functions of the KP-CSs for the PHO

Owing to the Wigner functions of quantum states can characterize the non-classical nature of the corresponding states, in this section we shall study the Wigner functions of the KP-CSs for the PHO. In the coherent state representation, the Wigner operator $\Delta(\alpha,\alpha^*)$ is [4]

$$\Delta(\alpha,\alpha^*)=\int\frac{d^2z}{\pi}\left|\alpha+z\right\rangle\left\langle\alpha-z\right|e^{\alpha z^*-\alpha^* z},\tag{7}$$

where $|\alpha+z\rangle$ is the coherent state [6, 7] and α is a complex number. By virtue of the technique of integration within an ordered product (IWOP) of operators [4], the Wigner operator $\Delta(\alpha,\alpha^*)$ is simplified as

$$\Delta(\alpha,\alpha^*)=\frac{2}{\pi}D(2\alpha)(-)^N,\tag{8}$$

where $N=a^+a$, $D(2\alpha)=\exp[2(\alpha a^+-\alpha^* a)]$ is the displaced operator.

For a pure state $|\psi\rangle$, we can express it in the following form

$$|\psi\rangle=\sum_{n=0}^{\infty}f_n\left|n\right\rangle,\tag{9}$$

its Wigner function is

$$W(\alpha,\alpha^*)=\langle\psi|\Delta(\alpha,\alpha^*)|\psi\rangle=\frac{2}{\pi}\sum_{m,n=0}^{\infty}f_m^*f_n(-1)^m\chi_{mn}(2\alpha),\tag{10}$$

where

$$\chi_{mn}(2\alpha)=\begin{cases}\sqrt{\frac{\Gamma(n+1)}{\Gamma(m+1)}}(2\alpha)^{m-n}\exp(-2\left|\alpha\right|^2)L_n^{m-n}(4\left|\alpha\right|^2) & m>n,\\ \sqrt{\frac{\Gamma(m+1)}{\Gamma(n+1)}}(-2\alpha^*)^{n-m}\exp(-2\left|\alpha\right|^2)L_m^{n-m}(4\left|\alpha\right|^2) & m<n,\end{cases}\tag{11}$$

where $L_n^m(x)$ denotes an associated Laguerre polynomials [14]

$$L_n^m(x)=\sum_{l=0}^{n}\binom{n+m}{n-l}\frac{(-x)^l}{l!}.\tag{12}$$

Further, the Wigner functions $W_{KP}(\alpha,\alpha^*)$ of the KP-CSs for the PHO can be expressed as a sum of the mixture part $W_{KP}^M(\alpha,\alpha^*)$ and the quantum interference part $W_{KP}^I(\alpha,\alpha^*)$ [13]

$$W_{KP}(\alpha,\alpha^*)=W_{KP}^M(\alpha,\alpha^*)+W_{KP}^I(\alpha,\alpha^*),\tag{13}$$

where the mixture part $W_{KP}^M(\alpha,\alpha^*)$ is found to be

$$W_{KP}^M(\alpha,\alpha^*)=\frac{2}{\pi}(1-|z|^2)^{2k}\exp(-2|\alpha|^2)\sum_{n=0}^{\infty}\frac{(-1)^n|z|^{2n}}{\frac{\Gamma(n+1)\Gamma(2k)}{\Gamma(n+2k)}}L_n(4|\alpha|^2)\tag{14}$$

and the quantum interference part $W_{KP}^I(\alpha,\alpha^*)$ has the following form

$$W_{KP}^I(\alpha,\alpha^*)=\frac{2}{\pi}(1-|z|^2)^{2k}\exp(-2\left|\alpha\right|^2)\sum_{\substack{m,n=0\\ m\neq n}}^{\infty}\frac{(-1)^m z^n z^{*m}}{\sqrt{\frac{\Gamma(n+1)\Gamma(2k)}{\Gamma(n+2k)}\frac{\Gamma(m+1)\Gamma(2k)}{\Gamma(m+2k)}}}$$

$$\times\left[\sqrt{\frac{\Gamma(n+1)}{\Gamma(m+1)}}(2\alpha)^{m-n}L_n^{m-n}(4|\alpha|^2)+\sqrt{\frac{\Gamma(m+1)}{\Gamma(n+1)}}(-2\alpha^*)^{n-m}\right.$$
$$\left.\times L_m^{n-m}(4|\alpha|^2)\right]. \tag{15}$$

By means of numerical computation the Wigner functions of the KP-CSs for the PHO are plotted for different values of parameters k and z against $\alpha = \mathrm{Re}\,\alpha + i\,\mathrm{Im}\,\alpha$ in Fig. 1. We would like to discuss the variations of the Wigner functions $W_{KP}(\alpha,\alpha^*)$ with different parameters k and z. We find that, for $|z| < 1$, the Wigner distributions have an almost Gaussian shape. When $k = 0.75$, from Fig. 1(a) we clearly see that the Wigner function $W_{KP}(\alpha,\alpha^*)$ displays only one prominent peak which is similar to the vacuum state. When $k = 0.75$, $z = 0.8$, the state $|z,k\rangle_{KP}$ is squeezed in the $\mathrm{Re}\,\alpha$ direction. Also we can see the peak begins to weaken by increasing z from Fig. 1(a) to (c). Especially, when the value of $|z|$ is almost equal to 1, the peak reduces to the lower position. Similarly, in Fig. 2, we plot the Wigner functions of the KP-CSs for the PHO with $z = 0.8$ for different values of $k = 3$

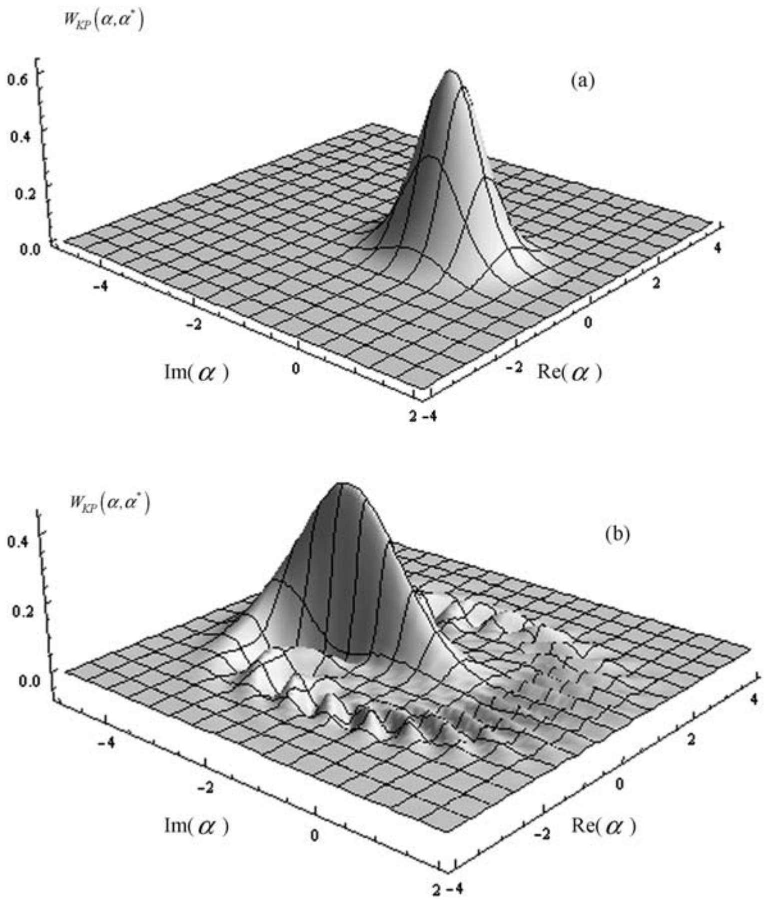

Fig. 1 Wigner functions of the KP-CSs for the PHO with α when (**a**) $k = 0.75$, $z = 0.3$; (**b**) $k = 0.75$, $z = 0.95$

and 7. From Fig. 2 we can clearly see that the peak of the Wigner function displaces from the center to the marginal position at the origin of phase space and the peak value reduces with increasing k. In a word, the greater the value of k, the smaller the value of the peak and the further the distance of the peak from the center position. In this case, the changes of the Wigner functions are very interesting. With the increasing k, the prominent peak is squeezed in the direction $\mathrm{Im}\,\alpha$ in comparison with the variations in Fig. 1 and the interference effect becomes stronger obviously. Also we can find in Figs. 1 and 2, the Wigner distributions are real functions which can demonstrate the nonclassical behavior of the KP-CSs for the PHO, respectively. In conclusion, the behavior of the Wigner functions $W_{KP}(\alpha,\alpha^*)$ is in agreement with the quantum features of the states $|z,k\rangle_{KP}$.

Now we want to compute the marginal distributions of the Wigner function $W_{KP}(\alpha,\alpha^*)$. We knew that the one-mode Wigner operator $\Delta(p,q)$ can be composed by virtue of IWOP technique, since its marginal integration is the position projector

$$\int_{-\infty}^{\infty} dp\Delta\left(p,q\right) = |q\rangle\left\langle q\right|, \tag{16}$$

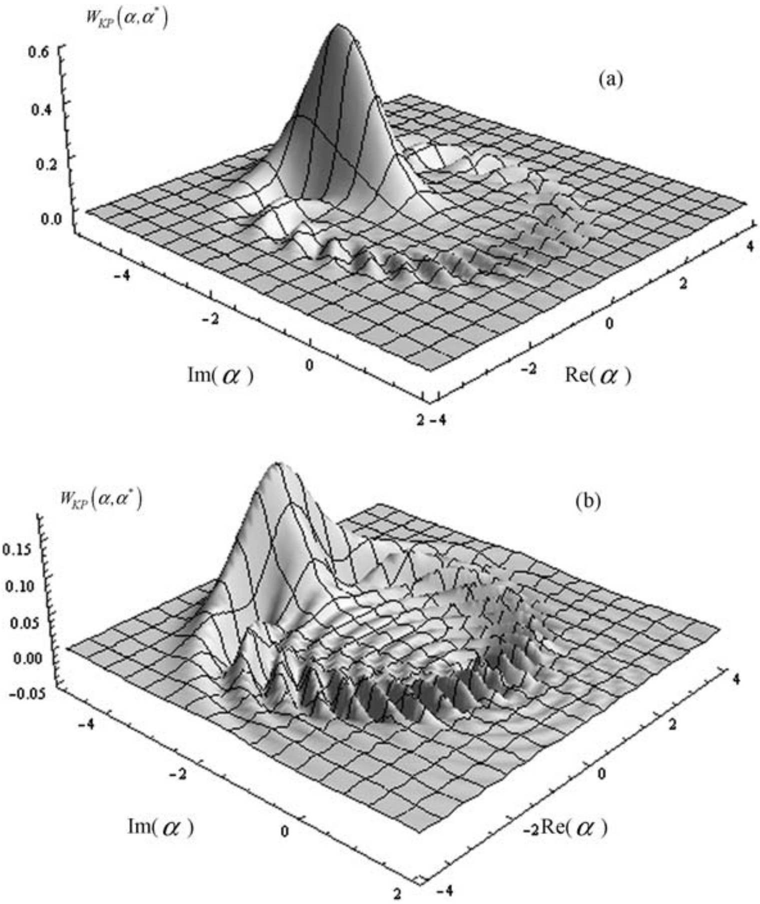

Fig. 2 Wigner functions of the KP-CSs for the PHO with α when (**a**) $z = 0.8$, $k = 3$; (**b**) $z = 0.8$, $k = 7$

and the momentum projector

$$\int_{-\infty}^{\infty} dq\,\Delta(p,q) = |p\rangle\langle p|, \tag{17}$$

where $|q\rangle$ and $|p\rangle$ are the standard eigenvectors of the coordinate operator Q and the momentum operator P, respectively.

If setting $\alpha = (q+ip)/\sqrt{2}$ in (14) and (15), then using (16) we can directly obtain the marginal distributions of the Wigner functions $W_{KP}(\alpha,\alpha^*)$ in the q direction

$$P(q) = \frac{1}{2\pi}\int dp W_{KP}(p,q) = \left|\langle q|\, z,k\rangle_{KP}\right|^2. \tag{18}$$

Due to the amplitude of $P(q)$

$$\begin{aligned}
\langle q|\, z,k\rangle_{KP} &= (1-|z|^2)^k \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{\frac{\Gamma(n+1)\Gamma(2k)}{\Gamma(n+2k)}}}\langle q|\, n\rangle \\
&= (1-|z|^2)^k \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{\frac{\Gamma(n+1)\Gamma(2k)n!}{\Gamma(n+2k)}}}\int dq' \langle q|a^{+n}|q'\rangle\langle q'|0\rangle \\
&= (1-|z|^2)^k \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{\frac{\Gamma(n+1)\Gamma(2k)n!}{\Gamma(n+2k)}}\sqrt{2^n\sqrt{\pi}}}\left(q-\frac{d}{dq}\right)^n \exp\left(-\frac{q^2}{2}\right) \\
&= (1-|z|^2)^k \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{\frac{\Gamma(n+1)\Gamma(2k)n!}{\Gamma(n+2k)}}\sqrt{2^n\sqrt{\pi}}} H_n(q)\exp\left(-\frac{q^2}{2}\right),
\end{aligned} \tag{19}$$

where we have used the formula for Hermite polynomials

$$H_n(q) = \exp\left(\frac{q^2}{2}\right)\left(q-\frac{d}{dq}\right)^n \exp\left(-\frac{q^2}{2}\right). \tag{20}$$

So we can obtain the marginal distributions in the q direction

$$P_{KP}(q) = \frac{(1-|z|^2)^{2k}\exp(-q^2)}{\sqrt{\pi}}\left|\sum_{n=0}^{\infty}\frac{z^n}{\sqrt{\frac{\Gamma(n+1)\Gamma(2k)n!}{\Gamma(n+2k)}}\sqrt{2^n}}H_n(q)\right|^2. \tag{21}$$

Similarly, using (14), (15) and (17), we can obtain another marginal distributions in the p direction

$$P_{KP}(p) = \frac{(1-|z|^2)^{2k}\exp(-p^2)}{\sqrt{\pi}}\left|\sum_{n=0}^{\infty}\frac{z^n}{\sqrt{\frac{\Gamma(n+1)\Gamma(2k)n!}{\Gamma(n+2k)}}\sqrt{2^n}}H_n(p)\right|^2. \tag{22}$$

From (21) and (22), we find that $P_{KP}(q)$ (or $P_{KP}(p)$) is the probability of finding particles with the given value q of the position (value p of the momentum) in the state $|z,k\rangle_{KP}$ given by (6). Therefore, the physical meaning of the Wigner function $W_{KP}(p,q)$ should also lie in its marginal distributions which gives the probability of finding the particles in the whole space.

3 Tomograms of the KP-CSs for the PHO

In recent year, the use of tomograms in quantum mechanics and quantum optics provides the possibility of describing a quantum state with a positive probability distribution. A direct description of quantum states by means of quantum tomograms for the system observables is interesting from both the theoretical and experimental points of view. Therefore, tomogram approach has brought much interest of some physicists. In this section we derive the tomograms of $|z,k\rangle_{KP}$ which are defined as

$$T_{KP}(q,f,g)=\iint\limits_{\infty} dp'dx'\delta(q-fq'-gp')W_{KP}(q',p'), \tag{23}$$

where $W_{KP}(q',p')$ is the Wigner functions for the states $|z,k\rangle_{KP}$. However, it will be very difficult to evaluate the integration if we directly substitute (14), (15) into (23). Luckily, we can use the following Radon transform between the Wigner operator and the projection operator $|q\rangle_{f,g\ f,g}\langle q|$ of intermediate coordinate-momentum state [5]

$$|q\rangle_{f,g\ f,g}\langle q|=\iint\limits_{\infty} dp'dx'\delta(q-fq'-gp')\Delta(q',p'). \tag{24}$$

Thus the tomograms of the states $|z,k\rangle_{KP}$ is

$$\begin{aligned}T_{KP}(q,f,g)&=\iint\limits_{\infty} dp'dx'\delta(q-fq'-gp')_{KP}\langle z,k|\,\Delta(q',p')\,|z,k\rangle_{KP}\\&=\left|{}_{f,g}\langle q\,|z,k\rangle_{KP}\right|^2,\end{aligned} \tag{25}$$

which shows that for tomogram approach there exists intermediate coordinate-momentum state $|q\rangle_{f,g}$, and the Radon transform of the Wigner operator is just the pure-state matrices $|q\rangle_{f,g\ f,g}\langle q|$. As a result, the tomogram of quantum states can be considered as the module-square of the states' wave function in intermediate coordinate-momentum representation. Using the completeness relation of coherent states, we can obtain the tomogram amplitude of the states $|z,k\rangle_{KP}$

$$\begin{aligned}{}_{f,g}\langle q\,|z,k\rangle_{KP}&={}_{f,g}\langle q|\int\frac{d^2\alpha}{\pi}\,|\alpha\rangle\,\langle\alpha\,|z,k\rangle_{KP}\\&=[\pi(f^2+g^2)]^{-1/4}(1-|z|^2)^k\sum_{n=0}^{\infty}\frac{z^n}{\sqrt{\frac{\Gamma(n+1)\Gamma(2k)}{\Gamma(n+2k)}}n!}\frac{\partial^n}{\partial\lambda^n}\int\frac{d^2\alpha}{\pi}\\&\quad\times\exp\left[-|\alpha|^2-\frac{q^2}{2(f^2+g^2)}+\frac{\sqrt{2}q}{f+ig}\alpha-\frac{f-ig}{2(f^2+g^2)}\alpha^2+\lambda\alpha^*\right]\Bigg|_{\lambda=0}\\&=[\pi(f^2+g^2)]^{-1/4}(1-|z|^2)^k\sum_{n=0}^{\infty}\frac{z^n}{\sqrt{\frac{\Gamma(n+1)\Gamma(2k)}{\Gamma(n+2k)}}n!}\\&\quad\times\left[\sqrt{\frac{f-ig}{2(f+ig)}}\right]^n H_n\left(\frac{q}{\sqrt{f^2+g^2}}\right)\exp\left[-\frac{q^2}{2\left(f^2+g^2\right)}\right],\end{aligned} \tag{26}$$

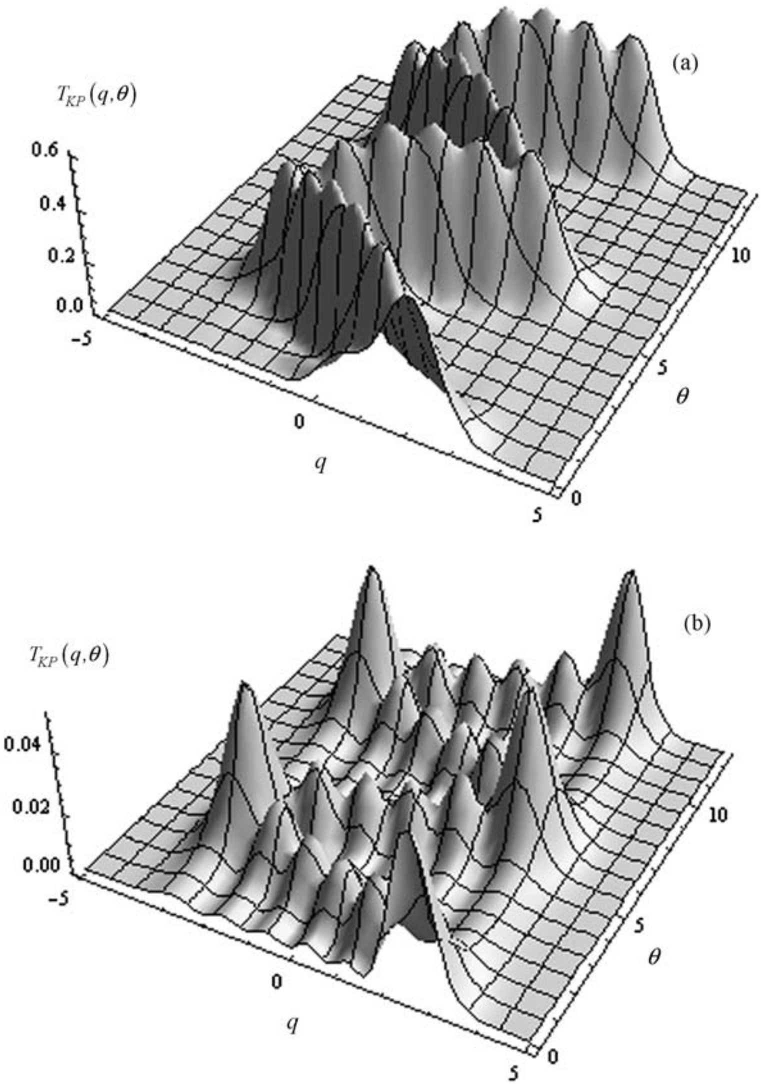

Fig. 3 Tomographic maps of the KP-CSs for the PHO with q, θ when (**a**) $z = 0.75$, $k = 0.75$; (**b**) $z = 0.75$, $k = 5$

where we have used the following integral formula

$$\int \frac{d^2 z}{\pi} \exp[\zeta\, |z|^2 + \xi z + \eta z^* + f z^2 + g z^{*2}] = \frac{1}{\sqrt{\zeta^2 - 4fg}} \exp\left[\frac{-\zeta\xi\eta + \xi^2 g + \eta^2 f}{\zeta^2 - 4fg}\right], \tag{27}$$

$$\mathrm{Re}\,(\zeta \pm f \pm g) < 0, \qquad \mathrm{Re}\left(\frac{\zeta^2 - 4fg}{\zeta \pm f \pm g}\right) < 0 \tag{28}$$

and the generating function of $H_n(q)$ is

$$H_n(q) = \frac{d^n}{dq^n} \exp\left(2qt - q^2\right)\Big|_{t=0}. \tag{29}$$

Thus we can obtain the tomograms of the states $|z, k\rangle_{KP}$

$$T_{KP}(q, f, g) = [\pi(f^2 + g^2)]^{-1/2}(1 - |z|^2)^{2k} \exp\left[\frac{-q^2}{f^2 + g^2}\right]$$
$$\times \left|\sum_{n=0}^{\infty} \frac{z^n \left[\frac{f-ig}{2(f+ig)}\right]^n}{\sqrt{\frac{\Gamma(n+1)\Gamma(2k)}{\Gamma(n+2k)}n!}} H_n\left(\frac{q}{\sqrt{f^2 + g^2}}\right)\right|^2. \tag{30}$$

In Fig. 3 we plot the tomograms $T_{KP}(q, \theta) = T_{KP}(q, f = \cos\theta, g = \sin\theta)$, of the KP-CSs for the PHO versus (q, θ) for indicated values of the parameters k and z. Tomograms are real and positive distribution functions. As one can see, the tomograms being a function of α are similar to the Q functions of the corresponding states. From Fig. 3, with the variation of the coordinate, the tomograms of the KP-CSs for the PHO present periodic variation. Furthermore, we clearly see that tomographic distributions shrink in the phase space with increasing k. When $z = 0.75$, we can find that the tomograms begin to appear multi-peak structure and with increasing the value k the peaks become more obvious, but the peak values decrease in varying degrees.

4 Conclusions

In summary, we have used the coherent state representation of Wigner operator and the intermediate coordinate-momentum representation to discuss the Wigner functions and the tomograms of the KP-CSs for the PHO. It is found that the variations of the Wigner function distribution with the parameters k and z (the tomogram with the parameters q and θ) reveal well interference properties of the KP-CSs for the PHO. The results of the tomograms for the KP-CSs for the PHO obtained by using the projection operator of the intermediate coordinate-momentum state $|q\rangle_{f,g\ f,g}\langle q|$ may be useful for experiments as references. Therefore, experimentally one can measure the module-square of the wave function $|z, k\rangle_{KP}$ in coherent state representation, then the tomograms for the KP-CSs for the PHO are obtained and also the states $|z, k\rangle_{KP}$ are measured.

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